Finding Complete Conformal Metrics to Extend Conformal Mappings

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1 Introduction

Let f be analytic in the unit disk $\mathbf{D}$, and let $Sf = (f''/f')' - (1/2)(f''/f')^2$ be its Schwarzian derivative. Starting with the fundamental work of Nehari, [11], and ever since the paper of Ahlfors and Weill, [2], univalence criteria involving the Schwarzian derivative have gone hand-in-hand with the phenomenon of quasiconformal extension to the sphere. The neatest formulation of this is a theorem of Gehring and Pommerenke, [9], which we state as follows:

Theorem 1 *Let $\rho(z)$ be any positive function on $\mathbf{D}$. Suppose that $|Sf(z)| \leq a\rho(z)^2$ implies f is univalent in $\mathbf{D}$. If $0 \leq t < 1$ and $|Sf(z)| \leq ta\rho(z)^2$ then f has a quasiconformal extension to $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$.*

The Gehring-Pommerenke result is actually more general than this in that it holds for functions whose domain is a quasidisk, *i.e.*, the image of $\mathbf{D}$ under a quasiconformal mapping of $\widehat{\mathbf{C}}$. On account of this generality their proof does not provide an explicit extension, and just such a formula is one of the most striking aspects of the Ahlfors-Weill paper. Ahlfors [1] does exhibit a quasiconformal extension for functions defined in a quasidisk and having small Schwarzian, modeled on the one used in [2], but his hypothesis is in terms of the Poincaré metric and not an arbitrary function ρ; see also [10].

A rich class of examples to which Theorem 1 can be applied comes from an older result of Nehari [12], [13]:

Theorem 2 (Nehari's p-criterion) *The function f will be univalent in $\mathbf{D}$ if*

$$|Sf(z)| \leq 2p(|z|), \tag{1.1}$$

where $p(x)$ is a function with the following properties: (a) $p(x)$ is positive and continuous for $-1 < x < 1$; (b) $p(-x) = p(x)$; (c) $(1-x^2)^2 p(x)$ is nonincreasing for $0 \leq x < 1$; (d) the differential equation

$$y'' + py = 0 \tag{1.2}$$

has a solution which does not vanish for $-1 < x < 1$.

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This includes many of the known results. For example, the choices of $p(x) = 1/(1-x^2)^2$ and $p(x) = \pi^2/4$ give Nehari's original univalence criteria in [11]:

$$|Sf(z)| \le \frac{2}{(1-|z|^2)^2}, \quad |Sf(z)| \le \frac{\pi^2}{2}. \tag{1.3}$$

There is now a fairly satisfactory treatment of homeomorphic and quasiconfomal extensions for functions satisfying $|Sf(z)| \le 2/(1-|z|^2)^2$ (the Nehari class); in addition to the original Ahlfors-Weill paper see [10], [5], and [8]. We will specifically exclude this case of the p-criterion in the present paper, and, interestingly, this is what allows our analysis of the other cases to proceed.

Recent progress in understanding the differential geometry associated with univalence criteria has emphasized the role of conformal metrics, the necessary distinctions between complete and incomplete metrics that then arise, and properties of extremal functions for the criteria, see [14], [4], [7]. Let $e^\sigma|dz|$ be a smooth conformal metric on $\mathbf{D}$. Let $0 < \delta \le \infty$ be the diameter of $\mathbf{D}$ with respect to this metric. The general univalence criterion in [14] involves both δ and the Gaussian curvature of the metric, and can be written in the form

$$|Sf - 2(\sigma_{zz} - \sigma_z^2)| \le 2\sigma_{z\bar{z}} + \frac{2\pi^2}{\delta^2}e^{2\sigma}, \tag{1.4}$$

see [4] and [7]. If an analytic function satisfies (1.4) it is univalent. The criterion comes from defining a generalized Schwarzian of the function f in a way that depends on the metric, see [15]; a conformal change of the metric changes the terms that appear in the criterion. When the metric is complete then $\delta = \infty$ and the condition becomes

$$|Sf - 2(\sigma_{zz} - \sigma_z^2)| \le 2\sigma_{z\bar{z}}. \tag{1.5}$$

As examples, to recover the conditions (1.3) from (1.5) and (1.4) take the metric $e^\sigma|dz|$ to be, respectively, the Poincaré metric (curvature -4, diameter ∞; complete), and the euclidean metric (curvature 0, diameter 2; incomplete).

Can we get from (1.1) to (1.4) or (1.5)? What is the metric? Nehari's original insight into the relationship bewteen univalence and differential equations was to relate the growth of the Schwarzian to the disconjugacy of solutions of (1.2) via Sturm comparison theorems. We turn this around. From a non-vanishing solution of (1.2) we construct a complete, radial, conformal metric of negative curvature for which (1.1) implies (1.5). This is done in Section 2 through the notion of an *extremal metric* for a p-criterion.

When a function satisfies (1.5), so for a complete metric, techniques and results are available that do not apply when the metric is incomplete. Thus in Section 3 we employ a generalization of the Ahlfors-Weill extension, as developed in [7], to obtain a *homeomorphic* extension to $\widehat{\mathbf{C}}$ for a function satisfying (1.1) strictly, $|Sf(z)| < 2p(|z|)$.

This extension may in some cases fail to be quasiconformal even when the strong bound holds, $|Sf(z)| \le 2tp(|z|)$, $t < 1$, and in overcoming this difficulty we were led to a general method for perturbing the metric. The analysis involves some technical aspects which may be of independent interest, and which we treat fully in Section 4. Furthermore, the resulting theorem proves to be more general than one would expect from Theorem 1, in that the Ahlfors-Weill formula based on the *perturbed metric* actually gives a quasiconformal extension for functions satisfying (1.1) strictly. This is Theorem 4 in Section 3.

We end this paper with a brief section on the criterion $|Sf(z)| \le \pi^2/2$ as one specific illustration of what can be done with the results here. None of the earlier techniques apply to constructing extensions for functions satisfying this very classical, very simple condition.

We hope that the ideas we present can be of help in understanding more general phenomena. For example, for what functions ρ can one prove a univalence result for a quasidisk as in the hypotheses of the Gehring-Pommerenke theorem? Is there a geometric way of capturing the properties of the function p in Theorem 2 that applies to the more general setting, and does the 'strict' versus 'strong' phenomenon hold there?

2 Extremal Radial Metrics and Perturbations

Radial Metrics from ODEs Let $p(x)$ be a nonnegative, continuous function on $[0, 1)$. From now on we suppose that the solution u of

$$u'' + pu = 0, \quad u(0) = 1, \, u'(0) = 0 \tag{2.1}$$

is positive. Consider the radial metric on $\mathbf{D}$ given by

$$|dw| = u^{-2}(|z|)|dz|, \tag{2.2}$$

and let

$$F(r) = \int_0^r u^{-2}(x)\, dx,$$

so that the metric can also be written as

$$F'(|z|)|dz|.$$

The diameter of the disk is then

$$\delta = 2F(1),$$

which may be infinite. The Gaussian curvature of the metric is

$$K(z) = -2u^4(|z|)(Au(|z|) + p(|z|)), \tag{2.3}$$

where

$$Au(|z|) = \left(\frac{u'}{u}(|z|)\right)^2 - \frac{1}{|z|}\frac{u'}{u}(|z|). \tag{2.4}$$

The initial conditions on u imply that Au is continuous at the origin, and that the curvature is negative.

The initial conditions also imply that u is decreasing, and hence that $\lim_{x\to 1^-} u(x)$ exists. In many cases this limit is zero and it matters to what order it vanishes. This determines whether $F(1)$ is infinite, *i.e.*, whether the metric is complete.

Extremal Radial Metrics, Extremal Functions, and Jordan Domains So far the only special aspect of the construction of the radial metric $F'(|z|)|dz|$ is the existence of the positive solution to the initial value problem (2.1). The function F is the solution to $SF = 2p$ on $[0, 1)$, normalized by $F(0) = 0$, $F'(0) = 1$ and $F''(0) = 0$.

From now on we assume the three hypotheses on p in Theorem 2: (a) $p(x)$ is positive and continuous for $-1 < x < 1$; (b) $p(-x) = p(x)$; (c) $(1 - x^2)^2 p(x)$ is nonincreasing for $0 \le x < 1$. When p is such a function we call the metric $F'(|z|)|dz|$ an *extremal radial metric* for the p-criterion, or, since we only work with radial metrics in this paper, simply an *extremal metric*. When referring to Theorem 2 we shall then speak of a 'p-criterion with an incomplete or complete metric'.

Under the stated hypotheses on p, F is defined only on $(-1,1)$. Many applications of Theorem 2 have an analytic p satisfying $|p(z)| \leq p(|z|)$ in the disk. In this case, if now

$$F(z) = \int_0^z u^{-2}(\zeta)\, d\zeta \tag{2.5}$$

then F will be analytic and univalent in $\mathbf{D}$. Of course, $F'(|z|)|dz|$ is also an extremal metric.

As introduced in [7], an *extremal function* satsifying the criterion (1.4), or (1.5), is a function whose image is not a Jordan domain. Since we will subsume Theorem 2 under the more general criterion, we also use this terminology in reference to a p-criterion. Still assuming that p is a analytic in $\mathbf{D}$, when $F(1) = \infty$ the function F is an extremal function since it is odd and so $F(1) = F(-1) = \infty$ as a point on the sphere. It was shown in [6] that, up to a rotation of F, this is the *only way* a function satisfying a p-criterion can fail to map $\mathbf{D}$ onto a Jordan domain. As a consequence, when p is not analytic all functions satisfying $|Sf(z)| \leq 2p(|z|)$ map onto Jordan domains, so there are no extremal functions. Moreover, in [3] it was shown that if $F(1) < \infty$ then $F(\mathbf{D})$ is actually a quasidisk, and so is $f(\mathbf{D})$ for any function satisfying $|Sf(z)| \leq 2p(|z|)$. Hence F is not an extremal function for the p-criterion, though $F'(|z|)|dz|$ is the corresponding (incomplete) extremal metric.

Complete Extremal Metrics and Nehari's p-criterion If $\sigma(z) = -2\log u(|z|)$, so the metric is also expressed as $e^{\sigma}|dz|$, then a straightforward calculation shows that the inequality (1.4) becomes, for $z \neq 0$,

$$|\zeta^2 Sf(z) + Au(|z|) - p(|z|)| \leq Au(|z|) + p(|z|) + \frac{2\pi^2}{\delta^2}u(|z|)^{-4}, \quad \zeta = \frac{z}{|z|}.$$

(This is a general calculation that holds for any conformal metric of the form $u^{-2}|dz|^2$.) In the complete case, which will be of most interest, this is

$$|\zeta^2 Sf(z) + Au(|z|) - p(|z|)| \leq Au(|z|) + p(|z|). \tag{2.6}$$

Via three lemmas we will show how a function f satisfying $|Sf(z)| \leq 2p(|z|)$ also satisfies (2.6). It is necessary to separate the cases when an extremal metric is complete or incomplete, that is, whether $F(1) = \infty$ or $F(1) < \infty$. We find that when the extremal metric is complete we obtain (2.6) directly, but when it is incomplete the criterion obtains for a completion of the extremal metric.

A helpful way of working with the distinction between complete and incomplete metrics comes from reworking the differential equation (2.1) to compare $p(x)$ to $1/(1-x^2)^2$ more directly. Write

$$q(x) = (1-x^2)^2 p(x).$$

Then $q(x) > 0$ and is nonincreasing on $[0,1)$.

If $q(0) \leq 1$ then $|Sf(z)| \leq 2p(|z|)$ would imply $|Sf(z)| \leq 2/(1-|z|^2)^2$. As we mentioned in the introduction, this is a situation that has been studied elsewhere, and by specifically excluding it here we are able to advance the understanding of the other univalence criteria included in Theorem 2. Thus for the rest of this paper we assume that

$$q(0) > 1.$$

The monotonicity of $q(x)$ implies that the limit $q(1) = \lim_{x\to 1^-} q(x)$ exists. The fact that u has no zeros implies that

$$q(1) < 1,$$

and it might be that $q(1) = 0$. To see why $q(1) < 1$, suppose by way of contradiction that $q(1) \geq 1$. Then $p(x) \geq 1/(1-x^2)^2$ and it follows from the Sturm comparison theorem that $u(x) \leq \sqrt{1-x^2}$. Hence

$$F(x) \geq L(x) = \frac{1}{2}\log\frac{1+x}{1-x},$$

for $x \in [0,1)$, and therefore F maps $(-1,1)$ onto $(-\infty,\infty)$. Consider the function $G(y) = F \circ L^{-1}(y)$. Note that $SL(x) = 2/(1-x^2)^2$ and recall that $SF(x) = 2p(x)$.

We can write

$$G(y) = \int_0^y v^{-2}(s)\, ds,$$

where

$$v'' + \frac{1}{2}(SG)v = 0, \quad v(0) = 1,\ v'(0) = 0.$$

Using the chain rule for the Schwarzian we see that $SG(y) \geq 0$, and not identically zero. (In fact, $SG(0) = 2(q(1)-1) \geq 0$.) Hence v is convex and not constant. This implies that for large s, $v(s) \geq as + b$, $a > 0$, which in turn implies that $G(\infty) < \infty$, a contradiction.

Next, u is a solution of (2.1) if and only if

$$w(x) = \frac{u(x)}{\sqrt{1-x^2}} \tag{2.7}$$

is a solution of

$$(1-x^2)((1-x^2)w'(x))' = (1-q(x))w(x), \quad w(0) = 1,\ w'(0) = 0.$$

The extra factor of $1-x^2$ in the derivative motivates the change of variable $x(s) = \tanh s$, for which $x' = 1-x^2$. Let

$$\varphi(s) = w(x(s)) \tag{2.8}$$

and

$$\nu(s) = q(x(s)). \tag{2.9}$$

Then $\varphi > 0$, ν is nonincreasing, and

$$\varphi''(s) = (1-\nu(s))\varphi(s), \quad \varphi(0) = 1,\ \varphi'(0) = 0. \tag{2.10}$$

φ is an even function so it suffices to analyze it for $s \geq 0$. At the end-points, $\nu(0) = q(0) > 1$ and $\nu(\infty) = q(1) < 1$. Hence φ is initally concave down, has an inflection point when $\nu = 1$ and then becomes convex. It may or may not have a critical point for $s > 0$, which, if it occurs, must be after ν becomes less than 1. Here is the key observation:

Lemma 1 $F(1) < \infty$ *if and only if* φ *has a critical point in* $(0,\infty)$.

Proof Suppose first that φ has no critical points. Then it is decreasing, and so too is u. Thus $w(x) \leq w(0) = 1$, that is,

$$u(x) \leq \sqrt{1-x^2}.$$

Therefore

$$F(r) = \int_0^r u^{-2}(x)\,dx \geq \int_0^r \frac{dx}{1-x^2} = \frac{1}{2}\log\frac{1+r}{1-r},$$

so F becomes infinite at 1, as in the argument above.

Now supose that φ does have a critical point in $(0,\infty)$. It follows from the differential equation (2.10) and the fact that ν is decreasing that for large s the solution φ will be bounded below by some exponential e^{ks}, $k > 0$. The resulting upper bound for u easily implies that F remains finite at 1.

In deriving (2.6) from $|Sf(z)| \leq 2p(|z|)$ when the extremal metric is complete, the main phenomenon is, essentially, an upper bound in terms of p for the curvature of the metric. It is more convenient for our present purposes to state this as an inequality between p and the operator Au, defined in (2.4). We have the following:

Lemma 2 *If the extremal metric $F'(|z|)|dz|$ is complete then $Au(|z|) \geq p(|z|)$.*

Proof Let $r = |z|$, and rewrite $Au(r) \geq p(r)$ as

$$ru^2(r)p(r) \leq r(u'(r))^2 - u(r)u'(r). \tag{2.11}$$

Suppose that $p(r)$ is C^1. Since both sides of (2.11) vanish at $r = 0$ it suffices to prove the inequality for the derivatives. After some cancellations using $u'' + pu = 0$, this is equivalent to

$$\frac{p'}{p} \leq -4\frac{u'}{u},$$

which in terms of q and w is

$$\frac{q'}{q} \leq -4\frac{w'}{w}.$$

But $q' \leq 0$, and since F is unbounded φ has no critical point so it is decreasing. Hence so is w, thus $w' \leq 0$ and the inequality is trivial. In the general case, when p is just continuous, simply approximate p uniformly on compact sets by smooth functions.

This is a rather simple, but useful result. Later we will need a parametrized version of the inequality that is more difficult.

Finally, we show that a p-criterion with an incomplete extremal metric can be embedded in a family of p-criteria where the final member has a complete extremal metric. This is based on the observation that, to an extent, one can scale the function p and still maintain all the properties.

Lemma 3 *Let p satisfy the hypotheses of Theorem 2 and suppose the associated extremal metric $F'(|z|)|dz|$ is incomplete. Then there exists a $1 < \tau_0 < \infty$ such that*
(i) For all $1 \leq \tau \leq \tau_0$ the functions τp satisfy the hypotheses of Theorem 2.
(ii) The associated extremal metrics $F'_\tau(|z|)|dz|$ are incomplete for $1 \leq \tau < \tau_0$, while $F'_{\tau_0}(|z|)|dz|$ is complete.

Proof By Lemma 1 the function φ, constructed from u (or F), has a critical point in $(0,\infty)$. Let $\tau > 1$ and consider u_τ,w_τ, q_τ, φ_τ and ν_τ corresponding to the function τp. Note that $q_\tau = \tau q$ and $\nu_\tau = \tau\nu$, so

$$\varphi''_\tau = (1-\tau\nu)\varphi_\tau.$$

Standard arguments from differential equations, based on continuous dependence on parameters, imply that there exists a largest value $\tau_0 > 1$ of the parameter τ such that the solution φ_τ is positive for all $\tau \le \tau_0$. Hence $u_\tau > 0$ for $1 \le \tau \le \tau_0$. It is easy to see that τ_0 is finite. This proves (i).

Next, the solution $\varphi_{\tau_0}(s)$ will necessarily be deceasing for all $s > 0$, while the φ_τ for $\tau < \tau_0$ will each have a critical point. This shows (ii).

We can now deduce:

Theorem 3 *If p satisfies the hypotheses of Theorem 2 and $|Sf(z)| \le 2p(|z|)$ then there is a complete extremal metric for which f satisfies (2.6).*

Proof As before, let $F'(|z|)|dz|$ be the associated extremal metric. Suppose first that the metric is complete. By Lemma 2,

$$\begin{aligned}|\zeta^2 Sf(z) + Au(|z|) - p(|z|)| &\le |Sf(z)| + Au(|z|) - p(|z|)\\ &\le 2p(|z|) + Au(|z|) - p(|z|) = Au(|z|) + p(|z|),\end{aligned}$$

which is (2.6).

Now suppose that the metric is incomplete. Appeal to Lemma 3 and simply observe that $|Sf(z)| \le 2p(|z|)$ trivially implies $|Sf(z)| \le 2\tau_0 p(|z|)$. The first part of the proof now applies to the complete extremal metric $F'_{\tau_0}(|z|)|dz|$.

Perturbing Complete Extremal Metrics In connection with the problem of quasiconformal extensions in the next section it will be necessary to perturb a complete extremal metric while still maintaining some of the key properties discussed above. We present the main lemmas here. The proofs are somewhat technical, and we defer the details till Section 4.

From Lemma 2 we know that $Au(|z|)/p(|z|) \ge 1$, but it will be important also to consider $\lim_{|z|\to 1} Au(|z|)/p(|z|)$ together with its relation to the nonincreasing function $q(x) = (1-x^2)^2 p(x)$ and its limit $q(1) = \lim_{x\to 1^-} q(x)$. The first result is as follows:

Lemma 4 *Suppose that the extremal metric $F'(|z|)|dz|$ is complete. Then*

$$\lim_{x\to 1^-}(1-x^2)\frac{u'}{u}(x) = -(1+\sqrt{1-q(1)}), \tag{2.12}$$

$$L = \lim_{|z|\to 1}\frac{Au(|z|)}{p(|z|)} = \lim_{|z|\to 1}\frac{1}{p(|z|)}\left(\frac{u'}{u}(|z|)\right)^2 = \frac{(1+\sqrt{1-q(1)})^2}{q(1)}. \tag{2.13}$$

Since $q(1) < 1$ we know that $L > 1$. It might be that $L = \infty$, equivalently that $q(1) = 0$. This happens, for example, with $p(x) = \pi^2/2$ or $p(x) = 2/(1-x^2)$. The latter choice of p produces the univalence criterion $|Sf(z)| \le 4/(1-|z|^2)$, see [13].

Handling the case $L = \infty$ presents problems in the next section. These are surmounted by perturbing the metric, a technique that works whether or not $L = \infty$.

It follows from (2.12) that

$$u(x) \sim \left(\frac{1-x}{1+x}\right)^{\beta}, \quad \beta = \frac{1}{2}(1+\sqrt{1-q(1)}) > \frac{1}{2}, \tag{2.14}$$

as $x \to 1$. Therefore, for $\alpha \in (1/2\beta, 1)$ the function

$$F_\alpha(r) = \int_0^r u^{-2\alpha}(x)\, dx$$

will be unbounded and define a complete radial metric

$$F'_\alpha(|z|)|dz|. \tag{2.15}$$

We refer to this as a *perturbed extremal metric.* Computing the Gaussian curvature as in (2.3) introduces quantities $A_\alpha u$ and p_α corresponding to Au and p, where now

$$A_\alpha(|z|) = \alpha^2 \left(\frac{u'}{u}(|z|)\right)^2 - \frac{\alpha}{|z|}\frac{u'}{u}(|z|) \tag{2.16}$$

$$p_\alpha(x) = \alpha\left(p(x) + (1-\alpha)\left(\frac{u'}{u}(x)\right)^2\right). \tag{2.17}$$

The inequality (2.6) becomes

$$|\zeta^2 Sf(z) + A_\alpha u(|z|) - p_\alpha(|z|)| \le A_\alpha u(|z|) + p_\alpha(|z|), \quad \zeta = \frac{z}{|z|}. \tag{2.18}$$

There is a version of Lemma 2 for perturbed metrics that includes the earlier result.

Lemma 5 *For a perturbed extremal metric there is an* $\alpha_0 \in (1/2\beta, 1)$ *such that* $A_\alpha u(|z|) \ge p_\alpha(|z|)$ *for all* $\alpha \in [\alpha_0, 1]$.

For our applications we have to get from a strict inequality $|Sf(z)| < 2p(|z|)$ in the unperturbed case to a strict inequality in (2.18) when the metric has been perturbed. For this we need:

Lemma 6 *If* $|Sf(z)| < 2p(|z|)$ *and* p *gives a complete extremal metric, then there exists an* $\alpha_1 \in (1/2\beta, 1)$ *such that* $|Sf(z)| < 2p_\alpha(|z|)$ *for all* $\alpha \in [\alpha_1, 1]$.

Now note that if $1 \ge \alpha > \max\{\alpha_0, \alpha_1\}$ and $|Sf(z)| < 2p(|z|)$ we obtain (2.18) by

$$\begin{aligned} |\zeta^2 Sf(z) + A_\alpha u(|z|) - p_\alpha(|z|)| &\le |Sf(z)| + A_\alpha(|z|) - p_\alpha(|z|) \\ &< A_\alpha(|z|) + p_\alpha(|z|). \end{aligned}$$

3 Quasiconformal Extensions

A function satisfying a p-criterion has a continuous extension to $\overline{\mathbf{D}}$. This was shown by the authors in [6] (and more generally in [7]) and by Steinmentz [16], [17]. We want to construct extensions to $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ for a function satisfying a strict p-criterion,

$$|Sf(z)| < 2p(|z|). \tag{3.1}$$

By virtue of the discussion of extremal functions in the preceding section, all functions satisfying (3.1) map $\mathbf{D}$ onto Jordan domains, at least.

According to Theorem 3, if a function f satisfies any p-criterion it then does so with a complete extremal metric. Thus we can now suppose without loss of generality that the extremal metric $F'(|z|)|dz|$ corresponding to p is complete.

We follow the construction in [7] of a generalization of the Ahlfors-Weill extension, applied in this case to $\mathbf{D}$ with a (possibly perturbed) extremal metric $F'_\alpha(|z|)|dz|$. Let $\Omega = f(\mathbf{D})$ and define a mapping of Ω by

$$\Lambda_f(w) = w + \frac{1}{\partial_w \log \rho_f(w)}, \tag{3.2}$$

where

$$\rho_f(f(z)) = \frac{F'_\alpha(|z|)}{|f'(z)|}.$$

Next define

$$E_f(z) = f(z) \text{for } |z| \leq 1, \Lambda_f(f(1/\bar{z})) \text{for } |z| > 1. \tag{3.3}$$

We remark that Λ_f is a conformally natural reflection, meaning that $\Lambda_{M\circ f} = M \circ \Lambda_f$ for any Möbius transformation M.

There are conditions discussed in [7], that will apply here, under which Λ_f is a reflection across $\partial\Omega$ and E_f is an extension of f. The fact that $f(z)$ and $\Lambda_f(f(1/\bar{z}))$ agree on $|z| = 1$ depends essentially on the fact that the metric is complete. We now state:

Theorem 4 *Suppose f satisfies Theorem 2 with a complete extremal metric. If $|Sf(z)| < 2p(|z|)$. then there exists an $\alpha \in (1/2\beta, 1]$ such that (3.3) is a quasiconformal extension of f to $\widehat{\mathbf{C}}$.*

First take $0 < \alpha \leq 1$ close enough to 1 so that Lemma 5 and Lemma 6 apply, and therefore so that

$$|\zeta^2 Sf(z) + A_\alpha u(|z|) - p_\alpha(|z|)| < A_\alpha u(|z|) + p_\alpha(|z|), \quad \zeta = z/|z|. \tag{3.4}$$

Using the results in [7], especially the proof of Corollary 5, this already implies that E_f is a homeomorphic extension of f. Briefly, if E_f is not a homeomorphism then there is some normalization of f by a Möbius transformation, $M \circ f$, for which $F'_\alpha(|z|)/|(M \circ f)'(|z|)|$ has at least two critical points. Then, by a convexity argument, this function attains its absolute minimum in $\mathbf{D}$ along the geodesic joining the two critical points. In turn, this forces equality to hold in (3.4), a contradiction.

Next, (see [7]) the Beltrami coeficient μ for the reflection Λ has magnitude

$$|(\mu \circ f)(z)| = \frac{|\zeta^2 Sf(z) + A_\alpha u(|z|) - p_\alpha(|z|)|}{A_\alpha u(|z|) + p_\alpha(|z|)}, \quad z \in \mathbf{D}. \tag{3.5}$$

By (3.4) we then know that $|\mu| < 1$. Thus the extension is quasiconformal away from any neighborhood of $\partial\mathbf{D}$. It is in estimating $|\mu \circ f|$ at the boundary that causes complications. For this we work with the quantity $L = \lim_{|z|\to 1} Au(|z|)/p(|z|)$ and separate the cases $L < \infty$ and $L = \infty$.

Suppose first that $L < \infty$. As a side comment we observe that if f satisfies the stronger condition

$$|Sf(z)| \leq 2tp(|z|),$$

for some $0 \leq t < 1$, as in the Gehring-Pommerenke theorem, then it is not necessary to perturb the metric to get a quasiconformal extension, *i.e.*, we can take $\alpha = 1$. For in estimating the Beltrami coefficient we then have

$$\begin{aligned} |(\mu \circ f)(z)| &\leq \frac{|Sf(z)| + |Au(|z|) - p(|z|)|}{Au(|z|) + p(|z|)} \\ &\leq \frac{Au(|z|) + (2t-1)p(|z|)}{Au(|z|) + p(|z|)} \to \frac{L + (2t-1)}{L+1} < 1, \end{aligned}$$

as $|z| \to 1$. Hence the extension E_f is quasiconformal everywhere.

We return to the assumption $|Sf(z)| < 2p(|z|)$. Choose α close enough to 1 so that $L\alpha > 1$; since $L > 1$ this is possible. For the Beltrami coefficient we have, using (2.12),

$$\begin{aligned} \frac{|\zeta^2 Sf(z) + A_\alpha u(|z|) - p_\alpha(|z|)|}{A_\alpha u(|z|) + p_\alpha(|z|)} &< \frac{2p(|z|) + A_\alpha(|z|) - p_\alpha(|z|)}{A_\alpha(|z|) + p_\alpha(|z|)} \\ &\to \frac{\alpha(2\alpha-1)L + (2-\alpha)}{\alpha L + \alpha}, \end{aligned}$$

as $|z| \to 1$. The last quantity is less than 1 precisely when $L\alpha > 1$, and again we conclude that the extension is quasiconformal everywhere.

Finally, suppose that $L = \infty$. Then the dominant terms in $A_\alpha u$ and p_α are those with $(u'/u)^2$. Hence this time for the Beltrami coefficient we can say

$$\begin{aligned} \frac{|\zeta^2 Sf(z) + A_\alpha u(|z|) - p_\alpha(|z|)|}{A_\alpha u(|z|) + p_\alpha(|z|)} &< \frac{2p(|z|) + A_\alpha(|z|) - p_\alpha(|z|)}{A_\alpha(|z|) + p_\alpha(|z|)} \\ &\sim \frac{\alpha(2\alpha-1)\left(\dfrac{u'}{u}(|z|)\right)^2}{\alpha\left(\dfrac{u'}{u}(|z|)\right)^2} = 2\alpha - 1 < 1 \end{aligned}$$

as $|z| \to 1$. The extension is therefore quasiconformal in this case too, and the theorem is proved.

Remarks on Homeomorphic and Quasiconformal Extension We stress that, on the one hand, even with an unperturbed metric (*i.e.*, taking $\alpha = 1$) the generalized Ahlfors-Weill extension is a homeomorphism of $\widehat{\mathbf{C}}$ when f satisfies the strict inequality $|Sf(z)| < 2p(|z|)$. On the other hand, without perturbing the metric the Ahlfors-Weill extension may not be quasiconformal even under the stronger assumption $|Sf(z)| \leq 2tp(|z|)$, $0 \leq t < 1$. The general phenomenon that a non-extremal function satisfying a p-criterion maps $\mathbf{D}$ onto a quasidisk can actually be deduced from another theorem of Gehring and Pommerenke, in the same paper [9], together with an extension of their work in [6], but without a method for obtaining the extension

There is, therefore, a gap in our understanding. Namely, while a non-extremal function for which $|Sf(z)| \leq 2p(|z|)$ is not satisfied strictly everywhere maps $\mathbf{D}$ onto a quasidisk, the analysis of the Ahlfors-Weill extension above does not apply. Nevertheless, Theorem 4 is optimal for the criterion $|Sf(z)| \leq \pi^2/2$, for example, since then, by the maximum principle, a non-extremal function must satisfy the criterion strictly. We discuss this example in Section 5.

4 Lemmas on Perturbed Extremal Metrics

In this section we prove Lemmas 4, 5, and 6 on perturbed extremal metrics.

Proof of Lemma 4 Lemma 4 is a statement on the values of two limits. For the first, we prove

$$\lim_{x\to 1^-}(1-x^2)\left|\frac{u'}{u}(x)\right| = 1+\sqrt{1-q(1)}. \tag{4.1}$$

Recall that u is decreasing and $u>0$. From (4.1) we then obtain a value for the limit $L = \lim_{|z|\to 1}(Au)|z|)/p(|z|))$ via

$$\begin{aligned}\frac{Au(|z|)}{p(|z|)} = \frac{(1-|z|^2)^2 Au(|z|)}{(1-|z|^2)^2 p(|z|)} &= \frac{(1-|z|^2)^2\left(\frac{u'}{u}(|z|)\right)^2 + \frac{(1-|z|^2)}{|z|}(1-|z|^2)\left|\frac{u'}{u}(|z|)\right|}{q(|z|)}\\ &\to \frac{(1+\sqrt{1-q(1)})^2}{q(1)}\end{aligned}$$

as $|z|\to 1$.

We also see from this that

$$L = \lim_{|z|\to 1}\frac{1}{p(|z|)}\left(\frac{u'}{u}(|z|)\right)^2, \tag{4.2}$$

the other exprression for L.

For (4.1), recall first the functions φ and ν in (2.8) and (2.9) and the differential equation $\varphi''(s) = (1-\nu(s))\varphi(s)$ in (2.10), where $x(s) = \tanh s$. We calculate that

$$(1-x(s)^2)\left|\frac{u'}{u}(x(s))\right| = -\frac{\varphi'}{\varphi}(s) + x(s), \tag{4.3}$$

and we shall prove that

$$\lim_{s\to\infty} -\frac{\varphi'}{\varphi}(s) = \sqrt{1-\nu(\infty)} = \sqrt{1-q(1)},$$

which, with (4.3), implies (4.1).

Let s_0 be the place at which φ has an inflection point, and fix any $s_1 > s_0$. Consider the two initial value problems

$$\begin{aligned}\phi''(s) &= (1-\nu(s_1))\phi(s), \quad \text{and}\\ \psi''(s) &= (1-\nu(\infty))\psi(s),\end{aligned}$$

where ϕ, ψ each agree to first order with φ at s_1. The solutions to these equations are just combinations of exponentials, of the form

$$\begin{aligned}\phi(s) &= ae^{\sqrt{1-\nu(s_1)}s} + be^{-\sqrt{1-\nu(s_1)}s},\\ \psi(s) &= ce^{\sqrt{1-\nu(\infty)}s} + de^{-\sqrt{1-\nu(\infty)}s}.\end{aligned}$$

Since ν is decreasing we must have $\varphi(s)\ge\phi(s)$. Then because φ is decreasing it must be that $a\le 0$, for otherwise the lower bound $\varphi\ge\phi$ would force φ to tend to infinity. This condition on a leads to

$$\sqrt{1-\nu(s_1)} \le -\frac{\varphi'}{\varphi}(s_1).$$

In similar fashion, $\varphi(s) \leq \psi(s)$, and also $c \geq 0$ for otherwise φ would eventually be negative. This condition on c leads to

$$-\frac{\varphi'}{\varphi}(s_1) \leq \sqrt{1-\nu(\infty)}.$$

Together these give

$$\sqrt{1-\nu(s_1)} \leq -\frac{\varphi'}{\varphi}(s_1) \leq \sqrt{1-\nu(\infty)},$$

and we conclude that

$$\lim_{s\to\infty} -\frac{\varphi'}{\varphi}(s) = \sqrt{1-\nu(\infty)} = \sqrt{1-q(1)},$$

as desired.

Proof of Lemma 5 We want to establish the inequality $A_\alpha u(|z|) \geq p_\alpha(|z|)$, where $A_\alpha u$ and p_α are defined in (2.16) and (2.17), and α is sufficiently close to 1. If we could show that $(1-x^2)^2 p_\alpha(x)$ is decreasing then this would follow as in Lemma 2. We have not been able to settle this one way or the other, so we must use a different argument. The main inequality, a local improvement of Lemma 2, is provided by the following lemma:

Lemma 7 *There exists an* $0 < r_0 < 1$ *such that*

$$Au(|z|) - p(|z|) \geq \frac{1}{2}p(0)(p(0)-1)|z|^2$$

for $|z| \leq r_0$.

Proof We recall the nonincreasing function $q(x) = p(x)(1-x^2)^2$, used many times to this point, and the functions $w(x) = u(x)/\sqrt{1-x^2}$ from (2.7) and φ from (2.8). Let $r = |z|$. Suppose that p is C^1. Then

$$(ru^2(r)(A(r)-p(r)))' = ru^2(r)p(r)\left(-4\frac{w'}{w}(r) - \frac{q'}{q}(r)\right) \geq -4ru^2(r)p(r)\frac{w'}{w}(r).$$

Hence

$$ru^2(r)(Au(r)-p(r)) \geq \int_0^r xu^2(x)p(x)\left(-\frac{w'}{w}(x)\right)dx = r\int_0^r \frac{xu^2(x)p(x)}{1-x^2}\left(-(1-x^2)\frac{w'}{w}(x)\right)dx.$$

But,

$$-(1-x^2)\frac{w'}{w}(x) = -\frac{\varphi'}{\varphi}(s), \quad s = \frac{1}{2}\log\frac{1+x}{1-x},$$

and $\varphi'' = (1-\nu)\varphi$. Since $\nu(0) > 1$, $\nu(s_0) > 1$ for $s_0 > 0$ small. Then $\nu(s) \geq \nu(s_0)$ for $s \in [0, s_0]$, and for s in this range the Sturm comparison theorem gives that

$$-\frac{\varphi'}{\varphi}(s) \geq -\frac{\eta'}{\eta}(s),$$

where η is the solution of

$$\eta'' = (1-\nu(s_0))\eta, \quad \eta(0) = 1,\ \eta'(0) = 0.$$

Thus $\eta(s)=\cos(\sqrt{\nu(s_0)-1}s)$ and

$$-\frac{\varphi'}{\varphi}(s)\geq\sqrt{\nu(s_0)-1}\tan(\sqrt{\nu(s_0)-1}s),$$

or

$$-(1-x^2)\frac{w'}{w}(x)\geq\sqrt{\nu(s_0)-1}\tan\left(\sqrt{\nu(s_0)-1}\frac{1}{2}\log\frac{1+x}{1-x}\right),$$

for $x\in[0,r_0]$, where $r_0=\tanh s_0$. Since $\nu(s_0)\to p(0)$ as $s\to 0$, it follows that for r_0 small enough and $x\in(0,r_0]$

$$-(1-x^2)\frac{w'}{w}(x)\geq\frac{1}{2}(p(0)-1)x.$$

Note that r_0 depends on the modulus of continuity of p at 0.

We conclude that for $r\in[0,r_0]$

$$ru^2(r)(Au(r)-p(r))\geq 4\int_0^r\frac{xu^2(x)p(x)}{1-x^2}\frac{1}{2}(p(0)-1)x\,dx\geq 2(p(0)-1)u^2(r)\int_0^r x^2p(x)\,dx,$$

because u is decreasing. Hence

$$ru^2(r)(Au(r)-p(r))\geq\frac{1}{2}p(0)(p(0)-1)u^2(r)r^3,$$

where, for $r\leq r_0$, small, we have used

$$\int_0^r x^2p(x)\,dx\geq\frac{3}{4}p(0)\int_0^r x^2\,dx=\frac{1}{4}p(0)r^3.$$

It follows that

$$Au(r)-p(r)\geq\frac{1}{2}p(0)(p(0)-1)r^2$$

for $r\in[0,r_0]$, where the number r_0 depends on the modulus of continuity of p at 0.

This proves the desired inequality if p is C^1. When p is C^0 we consider it as a uniform limit, say in the interval $[0,1/2]$, of C^1 functions.

Remark If p is C^1 then

$$ru^2(r)(Au(r)-p(r))=\int_0^r xu^2(x)p(x)\left(-4\frac{w'}{w}(x)-\frac{q'}{q}(x)\right)\,dx,$$

hence $ru^2(r)(Au(r)-p(r))$ is nondecreasing because the integrand is nonnegative. It follows that $ru^2(r)(Au(r)-p(r))$ remains nondecreasing when p is C^0. Since for $r>0$, small, $Au(r)-p(r)>0$ by the preceding lemma, we conclude that $Au(r)-p(r)>0$ for all $r>0$.

We can now prove Lemma 5 for perturbed extremal metrics, namely that

$$A_\alpha u(r)\geq p_\alpha(r),\quad r\in[0,1].$$

Suppose this is false. Then there is a sequence $\alpha_n\nearrow 1$ and points $r_n\in[0,1)$ such that $A_{\alpha_n}(r_n)<p_{\alpha_n}(r_n)$. As $r\to 1$ the leading term in A_α is $\alpha(u'/u)^2$ which will be bigger than the leading term $\alpha(1-\alpha)(u'/u)^2$ in p_α when α is close to 1, say $\alpha\geq\alpha_0$. This prohibits the r_n from accumulating

at 1. The preceding lemma also prohibits them from accumulating at 0. For this, note that the ineqaulity $A_\alpha u \geq p_\alpha$ is equivalent to

$$Au(r) - p(r) \geq 2(1-\alpha)\left(\frac{u'}{u}\right)^2(r).$$

Since $u'(0) = 0$, the lemma implies that $A_\alpha(r) \geq p_\alpha(r)$ for $r < r_0$ and $\alpha \geq \alpha_0$.

Therefore the r_n must accumulate at some $\bar{r} \in (0,1)$, which leads to

$$Au(\bar{r}) \leq p(\bar{r}).$$

This contradicts the Remark, above, and completes the proof of Lemma 5.

Proof of Lemma 6 The last lemma on perturbed metrics is the implication $|Sf(z)| < 2p(|z|) \implies |Sf(z)| < 2p_\alpha(|z|)$ for $\alpha \geq \alpha_1$, sufficiently close to 1. We recall the limit $L = \lim_{|z|\to 1} p(|z|)-1((u'/u)(|z|)^2$. Since $L > 1$ we have the inequality

$$p(|z|) \leq \alpha\left(\frac{u'}{u}(|z|)\right)^2, \tag{4.4}$$

for $|z|$ and α close enough to 1, respectively, say $r_1 \leq |z| < 1$ and $\alpha' \leq \alpha \leq 1$. After cancellation, (4.4) is precisely the statement that $p(|z|) \leq p_\alpha(|z|)$, and hence, on the one hand, $|Sf(|z|)| < 2p_\alpha(|z|)$ for $|z|$ and α in this range. On the other hand, since $|Sf(z)| < p(|z|)$ for all $z \in \mathbf{D}$ there is an $\alpha'' < 1$ so that $|Sf(z)| \leq 2\alpha p(|z| < 2p_\alpha(|z|)$ for $|z| \leq r_1$ and $\alpha'' \leq \alpha \leq 1$. These two estimates together prove the lemma for $\alpha_1 = \max\{\alpha', \alpha''\}$. Notice that α'', and hence α_1, depends on the size of $|Sf|$.

5 An Example

The univalence criterion $|Sf(z)| \leq \pi^2/2$, from Nehari's original 1949 paper, provides an interesting application of the ideas here. As mentioned in the introduction, it follows immediately from the general univalence criterion (1.4) with the simplest choice of background metric, the euclidean metric on $\mathbf{D}$. But the euclidean metric is not complete. If instead we use a p-criterion with $p(x) = \pi^2/4$ then the associated extremal metric is

$$F'(|z|)|dz| = \cos^{-2}\left(\frac{\pi}{2}|z|\right),$$

which *is* complete, and $|Sf(z)| \leq \pi^2/2$ implies

$$\left|\zeta^2 Sf(z) + \frac{\pi^2}{4}\tan^2\left(\frac{\pi}{2}|z|\right) + \frac{\pi}{2}\frac{1}{|z|}\tan\left(\frac{\pi}{2}|z|\right) - \frac{\pi^2}{4}\right| \leq \frac{\pi^2}{4}\tan^2\left(\frac{\pi}{2}|z|\right) + \frac{\pi}{2}\frac{1}{|z|}\tan\left(\frac{\pi}{2}|z|\right) + \frac{\pi^2}{4}.$$

Because $L = \infty$ in this case, we must still perturb the metric to obtain a quasiconformal extension for a function satisfying the strict inequality $|Sf(z)| < \pi^2/2$, or even the strong inequality $|Sf(z)| \leq 2t\pi^2/2$, $t < 1$. Observe that by the maximum principle, any non-extremal function satisfying $|Sf(z)| \leq \pi^2/2$ must satisfy the inequality strictly.

Refering first to (2.14), we need to find $\alpha \in (1/2, 1)$ so that Lemmas 5 and 6 hold for the metric $\cos^{-2\alpha}(|z|)|dz|$. The quasiconformal extension for f will then be induced by the reflection in the image given by

$$\Lambda_f(z) = f(z) + \frac{2|z|f'(z)}{\dfrac{\alpha\pi}{2}\tan\left(\dfrac{\pi}{2}|z|\right) - |z|\dfrac{f''}{f'}(z)}. \tag{5.5}$$

The inequality $A_\alpha u(|z|) \geq p_\alpha(|z|)$ for determining $\alpha \geq \alpha_0$ in Lemma 5 reduces to the requirement that

$$\alpha_0 \geq \frac{1}{2}\left[1 + \cot^2\left(\frac{\pi x}{2}\right)\left(1 - \frac{2}{\pi x}\tan\left(\frac{\pi}{2}x\right)\right)\right].$$

The maximum of the right hand side is 1/2, so any $\alpha \in (1/2, 1)$ will do here.

In Lemma 6 we need $\alpha \geq \alpha_1$ to get from $|Sf(z)| < \pi^2/2$ to the perturbed version

$$|Sf(z)| < p_\alpha(|z|) = \alpha\frac{\pi^2}{2}\left(1 + (1-\alpha)\tan^2\left(\frac{\pi}{2}x\right)\right). \tag{5.6}$$

Here, the choice of α_1 depends on the size of $|Sf|$, and so the parameter α in the reflection Λ_f will then depend on f. However, if we start with f satisfying the stronger inequality $|Sf(z)| \leq t\pi^2/2$ for some $0 \leq t < 1$ then (5.6) holds with any $\alpha \geq t$. In this case Λ_f will give a quasiconformal reflection in the image for any $\max\{1/2, t\} < \alpha < 1$.

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